Journal of Sound and Vibration (2002) **256**(3), 447–462
doi:10.1006/jsvi.2001.4241, available online at http://www.idealibrary.com on IDEAL®

EFFECTS OF BASE POINTS AND NORMALIZATION SCHEMES ON THE NON-LINEAR NORMAL MODES OF CONSERVATIVE SYSTEMS

X. H. ZHANG

Department of Engineering Mechanics, Xi'an Jiaotong University, Xi'an, Shaanxi Province 710049, People's Republic of China. E-mail: xhzhang@mail.xjtu.edu.cn

(*Received* 6 *November* 2000, *and in final form* 16 *November* 2001)

In this paper, two factors that affect the behaviors of the non-linear normal modes (NNMs) of conservative vibratory systems are investigated. The first factor is the base points (which are equivalent to Taylor series expanding points) of the non-linear normal modes and the second one is the normalization schemes of the corresponding linear modes. For non-linear systems, in general only the approximated NNM manifolds are obtainable in practice, so different base points may lead to different forms of NNM oscillators and different normalization schemes lead to different forward and backward transformations which in turn affect the numerical computation errors. Three different kinds of base points and two different normalization schemes are adopted for comparison respectively. Two examples of non-linear systems with two and three degrees of freedom, respectively, are given as illustration. Simulations for various cases are made. The analysis and the simulation results indicated that, the best base points are the abstract base points determined via the linear normal mode, which would eliminate the third order terms containing velocity (for cubic systems) or quadratic terms (for quadratic systems) in equations of the NNM oscillators. A better invariance of the NNMs would also be maintained with such base points. The best scheme of normalization is the norm-one scheme that would minimize the numerical errors.

1. INTRODUCTION

In the early 1960s, the concept of NNM was developed by Rosenberg [1] for undamped non-linear discrete vibratory systems. His work motivated much of interest in NNMs thereafter; especially in the past decade, a lot of research work has been done to study the behaviors of NNMs of non-linear vibratory systems by various methods. Essentially, these methods fall into two categories: one is the use of one-dimensional manifolds to approximate each of the NNMs and the other is the use of two (for non-internal resonant cases) or $2k$ (internal resonant cases)-dimensional manifolds to approximate each of the NNMs. For example, references [2–7] etc., belonged to the first category. Inspired by the theory of invariant manifolds of dynamical systems, Shaw and Pierre [8] proposed a new method which belongs to the second category. In their method, velocity as well as displacement of some mass point are used as independent variables to express the NNMs. This method is constructive locally and can be used to obtain the approximated NNMs of conservative as well as damped systems. Both similar and non-similar normal modes can be captured by the same procedure. For the applications of Shaw and Pierre's method, see also references [9–11]. In reference [12], Chechin *et al.* discussed the restrictions of the

0022-460X/02/$35.00

symmetries on the NNMs of the non-linear dynamical systems. For the NNMs of the continuous medium systems, see references [13–15].

In this paper, following the NNM methodology presented by Shaw and Pierre [8], the factors that influence the behaviors of NNMs of non-linear conservative vibratory systems are investigated. For linear vibratory systems, the normal modes can be obtained exactly by linear algebraic method. But for non-linear systems, in general, the exact closed-form NNMs could not be obtained, only approximated NNMs are obtained by Taylor series method. So the base points of NNMs, which correspond to the points about which the Taylor series are expanded, would affect the forms and behaviors of NNM oscillators. On the other hand, different normalization schemes of the corresponding linear normal modes would lead to different forward and backward transformations, which in turn affect the transfer relations of numerical errors. It is found that the best base points are the abstract points determined via the corresponding linear normal modes, and the best normalization scheme is the norm-one scheme. With such a choice, the third terms (for cubic systems) or quadratic terms (for quadratic systems) containing the velocity would not appear in equations of the NNMs oscillators. The qualitative properties of the non-linear modal systems are kept well in accordance with the original systems and a better invariance of the NNMs is maintained. At the same time, the numerical errors are not enlarged through the forward and backward transformations.

2. NON-LINEAR NORMAL MODE METHOD

For describing the method more concisely, an example is combined with illustration. The physical model to be considered here is a two-d.o.f. non-linear vibratory system [6, 9] as follows:

$$\begin{aligned} &\dot{x}_1 = y_1, \quad \dot{y}_1 = -x_1 - k(x_1 - x_2) - p_1 x_1^3 - q(x_1 - x_2)^3, \\ &\dot{x}_2 = y_2, \quad \dot{y}_2 = -(1+\alpha)x_2 - k(x_2 - x_1) - p_2 x_2^3 - q(x_2 - x_1)^3, \end{aligned} \tag{1}$$

where k, p_1, p_2, q, α are parameters.

Via the NNM method [8], the non-linear normal modes can be approximated by

$$\begin{aligned} x_i &= X_i(u, v) = a_{1i}u + a_{2i}v + \cdots + a_{9i}v^3 + \cdots, \\ y_i &= Y_i(u, v) = b_{1i}u + b_{2i}v + \cdots + b_{9i}v^3 + \cdots, \quad i = 1 \text{ or } 2, \end{aligned} \tag{2}$$

where (u, v) are the modal co-ordinates, in other words, they are the co-ordinates of the base points of the non-linear normal modes. The a_{ij}'s and b_{ij}'s are the NNMs' coefficients.

In vector form, the forward transformation (from modal co-ordinates to physical co-ordinates) is

$$z = M(w)w = \{M_0 + M_1(w) + M_2(w)\}w + \text{hot.}, \tag{3}$$

where, $z = (x_1, x_2, y_1, y_2)^{\mathrm{T}}$, $w = (u_1, v_1, u_2, v_2)^{\mathrm{T}}$, hot. = the higher order terms. $M(w)$ is assembled with the individual NNMs. It should be noted that the coefficients of the linear terms in equation (3) are equivalent to the linear modal coefficients.

The backward (inverse) transformation (from physical co-ordinates to modal co-ordinates) is

$$w = Q(z)z = \{Q_0 + Q_1(z) + Q_2(z)\}z + \text{hot.}, \tag{4}$$

where $Q(z)$ is the inverse of $M(w)$ in equation (3).

For linear systems, the linear normal modes (LNMs) can be obtained exactly. So different base points or/and normalization schemes of the LNMs would lead to the same results, but differently for non-linear systems, because only approximated NNMs can be obtained; so different base points of the NNMs or/and normalization schemes of the corresponding LNMs would lead to different results. Here the problems arise as to which base point should be chosen and which normalization scheme is the best one? In the following, we choose different base points and different normalization schemes for comparison.

According to the method of Shaw and Pierre, Xu *et al.* [9] obtained the non-linear normal modal equations of system (1) as follows:

$$\ddot{u}_i + (1 + k(1 - a_{1i}))u_i + (p_1 - a_{6i}k + q(1 - a_{1i})^3)u_i^3 - a_{8i}ku_i\dot{u}_i^2 = 0, \quad i = 1, 2 \tag{5}$$

where k, q, p_1 are parameters of the original physical systems, a_{1i}, a_{6i}, a_{8i} $(i = 1, 2)$ are the NNM coefficients. Note that, in reference [9], the expressions of the coefficients a_{6i}, a_{8i} $(i = 1, 2)$ have some minor mistakes and the correct formulas can be obtained easily, that are omitted here. When the parameters in equation (1) take the values $p_1 = g$, $p_2 = 0$, $q = 0$, $\alpha = 0$, model (1) degenerates to the example 2 of reference [8], and the corresponding NNM oscillators results as follows:

$$\text{mode I} \qquad \ddot{u}_1 + u_1 + gu_1\left[\left(1 + \frac{(3 - k)}{2(k - 4)}\right)u_1^2 + \frac{3}{2(k - 4)}\dot{u}_1^2\right] = 0,$$

$$\text{mode II} \qquad \ddot{u}_2 + (1 + 2k)u_2 + gu_2\left[\left(1 - \frac{(3 + 7k)}{2(4 + 9k)}\right)u_2^2 - \frac{3k}{2(4 + 9k)}\dot{u}_2^2\right] = 0. \tag{6}$$

In the above analysis, the base points are chosen as (x_1, y_1), and the normalization schemes are such that they set the first-component of each LNM to be one. For a convenient comparison, such base points are called fixed-component base point. Note that the NNMs discussed here are approximated to the third order only.

3. EFFECTS OF THE BASE POINTS ON THE NNMs

In contrast to the fixed-component base point of the NNMs chosen as above, two alternative choices are made as follows.

3.1. BASE POINTS VIA THE PRINCIPAL COMPONENT OF LNMs

The base point is chosen as $(u, v) = (x_I, y_I)$, i.e., in equation (2), set $a_{1I} = 1$, $a_{iI} = 0$ $(i = 2 \ldots 9)$, $b_{2I} = 1$, $b_{iI} = 0$ $(i = 1$ and $3, \ldots, 9)$, where $I = \text{subscript}\,(a_{1i})|\max\,(a_{1i}^2)$ $(i = 1, \ldots, N$, N is the number of d.o.f. of the system considered and a_{1i}'s are the linear modal coefficients). According to this selection rule, the base point would have the maximum component of LNM. When the point with the maximum component of LNM is not unique, then arbitrary one among them can be chosen as the base point, and the difference caused by different base points with the same maximum component of LNM is negligible. To illustrate the effect of this choice of base point on the NNMs, the FPU-model [16] with $N = 3$ are considered as follows:

$$\begin{aligned}
\dot{x}_1 &= y_1, \qquad \dot{y}_1 = x_2 - 2x_1 + \alpha((x_2 - x_1)^2 - x_1^2),\\
\dot{x}_2 &= y_2, \qquad \dot{y}_2 = x_1 + x_3 - 2x_2 + \alpha((x_3 - x_2)^2 - (x_2 - x_1)^2),\\
\dot{x}_3 &= y_3, \qquad \dot{y}_3 = x_2 - 2x_3 + \alpha(x_3^2 - (x_3 - x_2)^2).
\end{aligned} \tag{7}$$

Via the aforementioned condition, for mode-1 and mode-3, the base point is taken as (x_2, y_2), and for mode-2, the base point is taken as (x_1, y_1). The resulting NNM oscillators are as follows respectively:

$$\text{mode I} \quad \ddot{u}_1 + 0{\cdot}5858u_1 + \alpha^2 u_1[-1{\cdot}1082u_1^2 + 6{\cdot}7967\dot{u}_1^2] = 0,$$

$$\text{mode II} \quad \ddot{u}_2 + 2u_2 = 0,$$

$$\text{mode III} \quad \ddot{u}_3 + 3{\cdot}4142u_3 + \alpha^2 u_3[-2{\cdot}3204u_3^2 + 1{\cdot}0825\dot{u}_3^2] = 0. \tag{8}$$

Note: In transformation from equation (7) to (8), the corresponding NNM coefficients a_{ij}'s and b_{ij}'s can be easily obtained, so all are omitted here to save the space. In the rest of this paper, these coefficients are also omitted for the same reason.

On the other hand, if we choose the base point as (x_1, y_1) for all the three NNMs, the corresponding NNM oscillators would be as follows:

mode I

$$\ddot{u}_1 + 0{\cdot}5858u_1 + \alpha[-3{\cdot}4142\dot{u}_1^2 + 2{\cdot}2426u_1^2] + \alpha^2 u_1[-0{\cdot}612u_1^2 + 26{\cdot}2055\dot{u}_1^2] = 0,$$

mode II

$$\ddot{u}_2 + 2u_2 = 0,$$

mode III

$$\ddot{u}_3 + 3{\cdot}4142u_3 + \alpha[-0{\cdot}5858\dot{u}_3^2 + 6{\cdot}2426u_3^2] + \alpha^2 u_3[7{\cdot}4692u_3^2 + 2{\cdot}3659\dot{u}_3^2] = 0. \tag{9}$$

Obviously, the NNM oscillator equations in equation (8) are dramatically different from those in (9) and their forms appear very simple. These results indicate that, for quadratic non-linear vibratory systems, when the point with the max-component of LNM is chosen as the base point, the corresponding NNM oscillator equations would be very simple since the quadratic terms vanished.

3.2. BASE POINTS VIA THE LNMs

The base points via the LNMs are the abstract points (u, v) such that the physical co-ordinates can be expressed as $(x_i, y_i) = (a_{1i}u, b_{1i}v)$, where a_{1i} $(i = 1, \ldots, N)$ are the LNM coefficients. The LNMs are normalized to be norm-one. Here, two models are illustrated: model (1) and model (7). Since model (1) is simpler than model (7) and there were many results about it in the literatures, firstly it is reinvestigated in more detail as a comparison.

Via such a choice of the base points, the NNM oscillator equations of system (1) would take the following forms:

For modes I, II

$$\ddot{u}_i + c_{1i}u_i + c_{3i}u_i^3 = 0, \quad i = 1, 2, \tag{10}$$

where

$$c_{1i} = -(1 + k) \pm \tfrac{1}{2}\delta - \tfrac{1}{2}\alpha, \quad \delta = \sqrt{4k^2 + \alpha^2}. \tag{11}$$

c_{3i} are very complicated and given in Appendix B. When $i = 1$ equation (11) takes the " + " sign, $i = 2$ takes the " − " sign.

When the parameters in equation (1) take the values $p_1 = g$, $p_2 = 0$, $q = 0$, $\alpha = 0$, the corresponding non-linear normal modal equations are as follows:

$$\text{mode I} \quad \ddot{u}_1 + ku_1 + \frac{1}{4}u_1^3 g = 0,$$

$$\text{mode II} \quad \ddot{u}_2 + (2k+1)u_2 + \frac{1}{4}u_2^3 g = 0. \tag{12}$$

By comparison of the NNM oscillator equations (5) with equation (10), or equations (6) with equation (12), it can be found that the forms of the NNM oscillator equations in equations (10) or (12), which do not contain the velocity terms in it, are very simple. Another important difference between equations (6) and (12) is that, when the parameter $g > 0$, the first oscillator in equation (6) may bifurcate statically, but the oscillators in equation (12) cannot; in fact, even the original system cannot bifurcate statically. For NNM oscillators in equation (12), the corresponding modal amplitude ratios can be computed as follows:

$$\text{mode I} \qquad \frac{x_2}{x_1} = 1 + \frac{2(k-3)gu^2}{8k(k-4) - gu^2(k-3)}, \tag{13a}$$

$$\text{mode II} \qquad \frac{x_2}{x_1} = -1 + \frac{2(7k+3)gu^2}{-8k(9k+4) + gu^2(7k+3)}, \tag{13b}$$

where, unlike the results in equation [8], the modal amplitude ratios can be softening or hardening depending not only on the value of the coupling stiffness k but also on the non-linearity g and the amplitude u. For example, when $d_1 < 3 < d_2$, for the first mode (13a):

$k < d_1$, softening; $d_1 < k < 3$, hardening; $3 < k < d_2$, softening; $d_2 < k$, hardening;

where $d_i = 2 + \frac{1}{16}\left(gu^2 \mp \sqrt{1024 - 32gu^2 + g^2u^4}\right)$, $i = 1$ it takes the " $-$ " sign and $i = 2$ the " $+$ " sign.

Note that the boundary points d_1 and d_2 above are dependent on the amplitude u. As for other cases it can be discussed similarly. These results indicate that the choice of fixed-component base points may change the behaviors of the NNM oscillators and the base points via the LNMs may be a better one.

To verify this observation further, FPU model (7) is analyzed as follows, with the same choice of base points of NNMs as those in (10) and (12):

$$\begin{aligned} &\text{mode I} && \ddot{u}_1 + 0{\cdot}5858u_1 + \alpha^2 u_1[0{\cdot}2071u_1^2 - 0{\cdot}5\dot{u}_1^2] = 0,\\ &\text{mode II} && \ddot{u}_2 + 2u_2 = 0,\\ &\text{mode III} && \ddot{u}_3 + 3{\cdot}4142u_3 - \alpha^2 u_3[1{\cdot}2071u_3^2 + 0{\cdot}5\dot{u}_3^2] = 0. \end{aligned} \tag{14}$$

Through the comparison of equations (9) and (14), it can be found that the latter is much simpler than the former because the quadratic terms in it have all vanished. Through the comparison of NNMs in equations (8) and (14), it seems that the choice of the abstract points as the base points is not good. But after a more detailed investigation, it can be easily found that the non-linearity in the NNMs oscillators of model (14) is more weak than those of model (8). This means that model (14) is superior to model (8) in the further perturbative solutions of NNMs oscillators. Similar to the second oscillator in equation (8), the second NNM oscillator in (14) is also linear. This is a very interesting point that the NNM

oscillator presents itself as a linear one! In fact, this linear mode describes an exact modal motion of system (7). Another aspect that should be stressed is that, although the NNM oscillators in equation (14) possesses some symmetry, for example, they are invariant under the transformation that $u_i \to -u_i$, deduction cannot be made that the original physical model also possesses such a symmetry. This is because the transformations between the physical co-ordinates and modal co-ordinates are non-linear and do not possess such a symmetry.

4. EFFECTS OF THE NORMALIZATION SCHEMES ON THE NNMs

4.1. *I*TH-COMPONENT-ONE SCHEME

The first normalization scheme is considered here, it is set as $(x_i, y_i) = (u, v)$, i.e., $a_{1i} = 1$, $a_{2i} = 0$, $a_{Ji} = 0$ $(J = 3, \ldots, 9)$, $b_{1i} = 0$, $b_{2i} = 1$, $b_{Ji} = 0$ $(J = 3, \ldots, 9)$, i.e., the ith mass point is chosen as the base point.

If each of the first-component of the LNMs is set to be one and the base points are chosen via the LNMs, then the corresponding NNM oscillators of FPU model (7) are as follows:

$$\begin{aligned} &\text{mode I} && \ddot{u}_1 + 2{\cdot}3431u_1 - \alpha^2 u_1[0{\cdot}253u_1^2 + 0{\cdot}2547\dot{u}_1^2] = 0,\\ &\text{mode II} && \ddot{u}_2 + 4u_2 = 0,\\ &\text{mode III} && \ddot{u}_3 + 13{\cdot}6569u_3 - \alpha^2 u_3[12{\cdot}6601u_3^2 + 1{\cdot}0468\dot{u}_3^2] = 0. \end{aligned} \tag{15}$$

From equations (14) and (15), it can be found that the forms of the NNM oscillator equations are the same, only the coefficients are different. Similarly, the corresponding NNM manifolds are also different. To inspect the effects of normalization schemes on the NNMs further, the determinants of the Jacobian of the linear part (i.e., the linear modal matrix M_0) of the non-linear modal transformation (3) corresponding to various base points are computed and listed in Table 1.

For weakly non-linear systems, the determinants of the Jacobians of the NNM transformations (3), (4) can be viewed as the error-amplification factors. The determinants of the matrices M_0 and Q_0 can be used as a measure of the main part of the error-transfer relationships between the physical co-ordinates and the non-linear modal co-ordinates. Since the numerical errors may be enlarged not only by forward transformation but also by backward transformation, hence, the ideal normalization scheme is such that $\min(\text{abs}(\det(M_0)) + \text{abs}(\det(M_0^{-1})))$, i.e., $\text{abs}(\det(M_0)) = 1$. As the number of d.o.f. of the

TABLE 1

Determinants of the LNM matrix

No. of equations	Determinant
(5)	$-(\alpha^2 + 4k^2)/(k^2)$
(6)	-4
(8)	-8
(9)	-32
(10)	-1
(12)	-1
(14)	-1
(15)	-32

systems increases, the determinants corresponding to all the normalization schemes excluding the norm-one scheme would become dramatically large [10] or small.

4.2. NORM-ONE SCHEME

This normalization scheme is set as $\Sigma_i a_{1i}^2 = 1$, $\Sigma_i b_{2i}^2 = 1$, $(u, v) = (\Sigma_i a_{1i} x_i, \Sigma_i b_{2i} y_i)$, where, the a_{1i}'s and b_{2i}'s are the coefficients of the ith linear normal mode. The equations of the NNMs oscillators are given above as (10), (12), (14), etc., and the determinants of the corresponding LNM matrices have been listed in Table 1. From Table 1 it can be seen that, only with this normalization scheme the determinants are one in absolute value.

5. SIMULATION RESULTS AND REMARKS

For comparison of the effects of different base points on the invariance of the non-linear normal modal subspaces, model (1) and FPU-model (7) are investigated by simulation, respectively. Here, for each model, two different kinds of points, i.e., the fixed-component point and abstract point via the LNMs are chosen as the base points of the NNMs. In all cases, the simulations are over a time interval of [0, 20]. All the individual non-linear modal phase manifolds are obtained by projection of the trajectory of the original model for various initial conditions started on one of the NNMs with a specific energy. It should be noted that the initial modal amplitudes of the NNMs have the same value which does not mean that the original system would possess the same energy, in fact, they may have a very different energy. More precisely, all the comparisons here are based on the same initial energy instead of the same initial non-linear modal amplitude. The expression of the potential energy (PE) for system (1) is as follows:

$$PE = \frac{1}{2}x_1^2 + k\left(\frac{1}{2}x_1^2 - x_1x_2\right) + \frac{1}{4}p_1x_1^4 + \frac{1}{4}q(x_1 - x_2)^4 + \frac{1}{2}(1+\alpha)x_2^2 + k\left(\frac{1}{2}x_2^2 - x_1x_2\right) + \frac{1}{4}p_2x_2^4 + \frac{1}{4}q(x_2 - x_1)^4. \quad (16)$$

Similarly, the potential energy for system (7) is given as follows:

$$PE = x_1^2 + x_2^2 + x_3^2 - 2(x_1x_2 + x_2x_3) + \frac{1}{3}\alpha(2(x_2 - x_1)^3 + 2(x_3 - x_2)^3 + x_1^3 - x_3^3). \quad (17)$$

The specific values of the initial energy are taken here purely for convenience. All the simulation results are presented in Tables 2–4 for various cases, where PE_i means the potential energy of the ith NNM.

From Figure 1 and Table 2, it can be seen that the amplitudes and therefore the corresponding maximal energy of the non-initial-excited NNMs in (a) and (b) are much larger than those in Figures 1(c) and 1(d). These results indicated that the choice of abstract base point via the LNM is better than those of fixed-component base point.

From Figure 2 and Table 3, it can be found that the amplitudes and the corresponding maximal energy of the non-initial-excited NNMs in (e) and (f) are smaller than those in Figures 2(g) and 2(h). This may be caused by the asymmetry of the original system (i.e., $p_1 \neq p_2$). It is indicated that the fixed-component base point is superior to the abstract base point. For a thorough comparison more the cases approximated to fifth order are also simulated, see Figure 3 and Table 3; the results indicated that, these two choices of base

TABLE 2

Simulation results of the model (1) *with parameters* $k = 1$, $p_1 = p_2 = 0{\cdot}5$, $q = \alpha = 0{\cdot}3$

Approximated to third accuracy	Base point (x_1, y_1)	Base point $(\Sigma_i a_{1i} x_i, \Sigma_i b_{2i} y_i)$
	Figure 1(a)	Figure 1(c)
Initial conditions on the first NNM	(1, 0, 0·899, 0)	(0·9991, 0, 0·9004, 0)
Initial energy	$PE_1 = 0{\cdot}3381$	$PE_1 = 0{\cdot}3381$
Initial modal amplitude	$u_1 = 1{\cdot}0$	$u_1 = 1{\cdot}345$
	Figure 1(b)	Figure 1(d)
Initial conditions on the second NNM	(1, 0, $-$ 1·053, 0)	(0·9922, 0, $-$ 1·06, 0)
Initial energy	$PE_2 = 7{\cdot}32$	$PE_2 = 7{\cdot}32$
Initial modal amplitude	$u_2 = 1{\cdot}0$	$u_2 = 1{\cdot}451$

TABLE 3

Simulation results of the model (1) *with parameters* $k = 1$, $p_1 = 0{\cdot}5$, $p_2 = q = \alpha = 0$

Approximated to third accuracy	Base point (x_1, y_1)	Base point $(\Sigma_i a_{1i} x_i, \Sigma_i b_{2i} y_i)$
	Figure 2(e)	Figure 2(g)
Initial conditions on the first NNM	(1, 0, 7/6, 0)	(0·9697, 0, 1·175, 0)
Initial energy	$PE_1 = 0{\cdot}1528$	$PE_1 = 0{\cdot}1528$
Initial modal amplitude	$u_1 = 1{\cdot}0$	$u_1 = 1{\cdot}517$
Maximal energy of the second NNM	$PE_2 = 3{\cdot}89 \times 10^{-3}$	$PE_2 = 7{\cdot}44 \times 10^{-3}$
	Figure 2(f)	Figure 2(h)
Initial conditions on the second NNM	(1, 0, $-$ 21/26, 0)	(0·9768, 0, $-$ 0·834, 0)
Initial energy	$PE_2 = 3{\cdot}393$	$PE_2 = 3{\cdot}393$
Initial modal amplitude	$u_2 = 1{\cdot}0$	$u_2 = 1{\cdot}28$
Maximal energy of the first NNM	$PE_1 = 2{\cdot}8 \times 10^{-8}$	$PE_1 = 5{\cdot}7 \times 10^{-8}$
Approximated to third accuracy	Base point (x_2, y_2)	Base point $(\Sigma_i a_{1i} x_i, \Sigma_i b_{2i} y_i)$
	Figure 3(el)	Figure 2(g)
Initial conditions on the first NNM	(0·902, 0, 1·167, 0)	
Initial energy	$PE_1 = 0{\cdot}1528$	All as given above
Initial modal amplitude	$u_1 = 1{\cdot}167$	
Maximal energy of the second NNM	$PE_2 = 2{\cdot}55 \times 10^{-2}$	
	Figure 3(f1)	Figure 2(h)
Initial conditions on the second NNM	($-$ 0·9647, 0, 0·8476, 0)	
Initial energy	$PE_2 = 3{\cdot}393$	All as given above
Initial modal amplitude	$u_2 = 0{\cdot}8476$	
Maximal energy of the first NNM	$PE_1 = 2{\cdot}6 \times 10^{-7}$	
Approximated to fifth accuracy	Base point (x_2, y_2)	Base point $(\Sigma_i a_{1i} x_i, \Sigma_i b_{2i} y_i)$
	Figure 3(e[5])	Figure 3(g[5])
Initial conditions on the first NNM	(1, 0, 1·182, 0)	(1·004, 0, 1·18, 0)
Initial energy	$PE_1 = 0{\cdot}1581$	$PE_1 = 0{\cdot}1581$
Initial modal amplitude	$u_1 = 1{\cdot}0$	$u_1 = 1{\cdot}545$
Maximal energy of the second NNM	$PE_2 = 1{\cdot}4 \times 10^{-3}$	$PE_2 = 1{\cdot}8 \times 10^{-3}$
	Figure 3(f[5]	Figure 3(h[5])
Initial conditions on the second NNM	(1, 0, $-$ 0·8274, 0)	(0·9986, 0, $-$ 0·829, 0)
Initial energy	$PE_2 = 3{\cdot}464$	$PE_2 = 3{\cdot}464$
Initial modal amplitude	$u_2 = 1{\cdot}0$	$u_2 = 1{\cdot}292$
Maximal energy of the first NNM	$PE_1 < 10^{-8}$	$PE_1 < 10^{-8}$

points are almost identical. On the other hand, when the point (x_2, y_2) instead of (x_1, y_1) is chosen as the base point, the results showed that the abstract base point is much better than the fixed-component base point, see also Figure 3 and Table 3.

TABLE 4

Simulation results of model (7) *with parameters* $\alpha = 0{\cdot}25$

Approximated to third accuracy	Base point (x_1, y_1)	Base point $(\Sigma_i a_{1i}x_i, \Sigma_i b_{2i}y_i)$
	Figure 4(i)	Figure 4(l)
Initial conditions on the first NNM	(1, 0, 1·172, 0, 1·172, 0)	(1·076, 0, 1·21, 0, 0·6967, 0)
Initial energy	$PE_1 = -1{\cdot}13$	$PE_1 = -1{\cdot}13$
Initial modal amplitude	$u_1 = 1{\cdot}0$	$u_1 = 1{\cdot}742$
	Figure 4(j)	Figure 4(m)
Initial conditions on the second NNM	(1, 0, 0, 0, −1, 0)	(−0·9226, 0, 0, 0, 0·9226, 0)
Initial energy	$PE_2 = 1{\cdot}833$	$PE_2 = 1{\cdot}833$
Initial modal amplitude	$u_2 = 1{\cdot}0$	$u_2 = -1{\cdot}305$
	Figure 4(k)	Figure 4(n)
Initial conditions on the third NNM	(1, 0, −1·1, 0, 0·75, 0)	(1·041, 0, −1·144, 0, 0·69, 0)
Initial energy	$PE_3 = 6{\cdot}186$	$PE_3 = 6{\cdot}186$
Initial modal amplitude	$u_3 = 1{\cdot}0$	$u_3 = 1{\cdot}674$

When approximated to fifth order, the non-linear modal oscillators of system (1) would be as follows:

$$\text{mode I} \qquad \ddot{u}_1 + u_1 + 0{\cdot}125u_1^3 - 0{\cdot}015625u_1^5 - 0{\cdot}0234375u_1^3\dot{u}_1^2 = 0,$$

$$\text{mode II} \qquad \ddot{u}_2 + 3u_2 + 0{\cdot}125u_2^3 + 0.018u_2^5 + 0{\cdot}0054u_2^3\dot{u}_2^2 = 0. \tag{18}$$

Note that the fifth order terms containing the velocity in (18) does not vanish. This might mean that some new base point should be chosen to eliminate the velocity terms.

To further investigate the effects of the base points on the invariance of the NNMs, model (7) with parameter $\alpha = 0{\cdot}25$ is simulated for various cases. See Figure 4 and Table 4. These results verified that the choice of abstract base point via the LNM is superior to the choice of fixed-component base point.

6. CONCLUSIONS

For the conservative vibratory systems, based on the analysis and simulation results above, our conclusions are as follows:

1. The forms of NNM oscillators depend dramatically on the choice of base points. The best base points may be those that are determined via the corresponding linear normal modes. With such base points, the qualitative properties of the non-linear modal system are maintained very well in accordance with the original system and a better invariance of the NNMs is also maintained.
2. Different normalization schemes would lead to different forward and backward transformations, which in turn lead to distinct error-transfer relationships. Especially for large d.o.f.s systems, the effects would be more remarkable. The best normalization scheme is the norm-one scheme.
3. It can be anticipated that both localized and non-localized modes could be treated in a universal way with the choice of abstract base point via the LNM. For damped vibratory systems, the normalization schemes would affect the NNMs in the same way as the one mentioned above. For continuous medium systems, the base points and normalization schemes would operate in the same way as in the discrete systems.

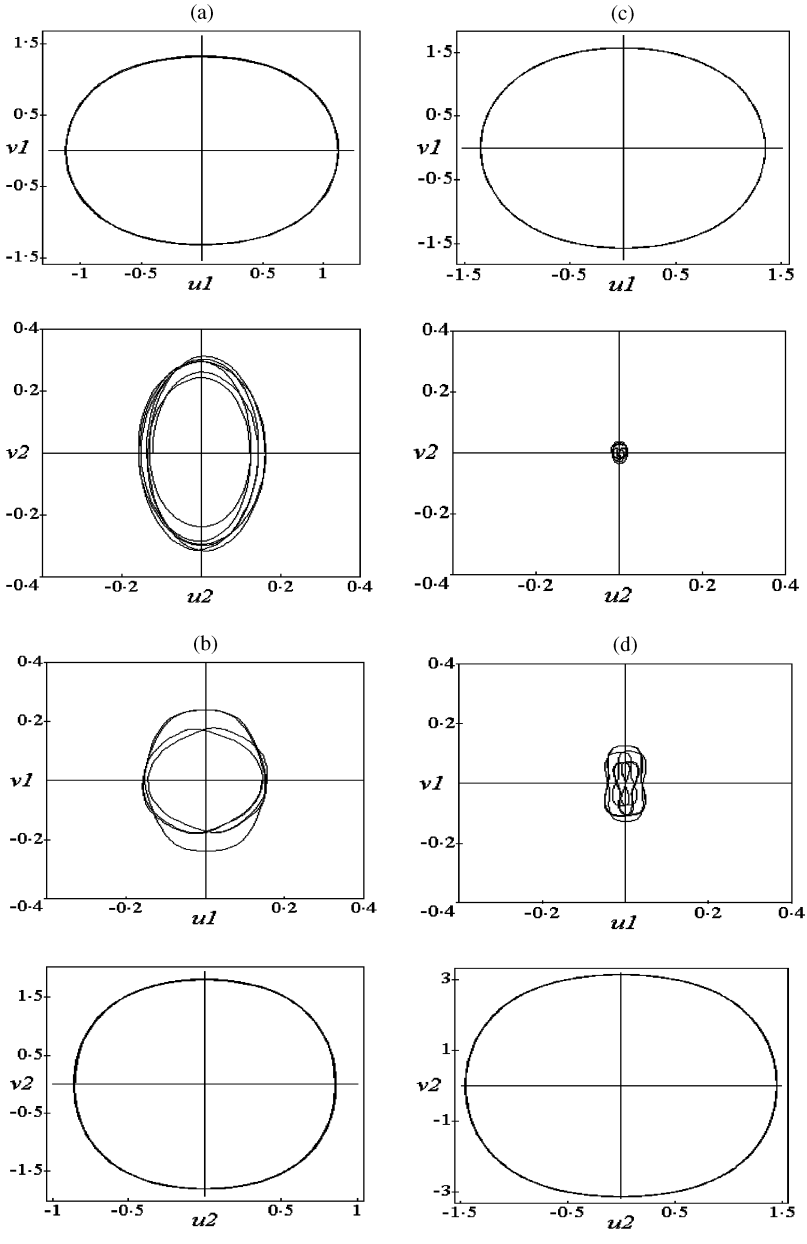

Figure 1. Effects of the base points on the invariance of the non-linear normal modal subspaces of the model (1) with the parameters $k = 1$, $p_1 = p_2 = 0{\cdot}5$, $q = \alpha = 0{\cdot}3$, studied by simulation; all of them are non-linear normal phase manifolds obtained by projection of the trajectory for different initial conditions, energy and base points: (a) [base point, initial condition, energy] = [first component point, first NNM, 0·3381]; (b) [base point, initial condition, energy] = [first component point, second NNM, 7·32]; (c) [base point, initial condition, energy] = [abstract point, first NNM, 0·3381]; (d) [base point, initial condition, energy] = [abstract point, second NNM, 7·32].

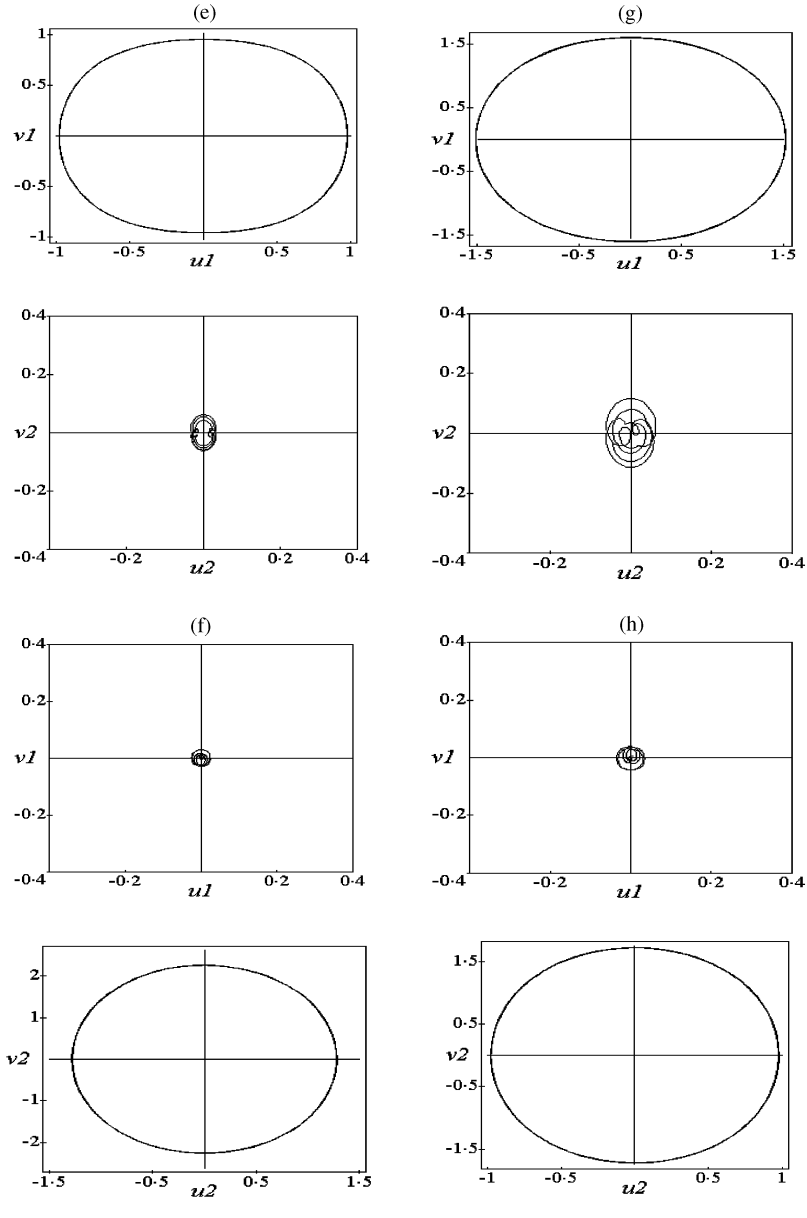

Figure 2. Effects of the base points on the invariance of the non-linear normal modal subspaces of the model (1) with the parameters $k = 1$, $p_1 = g = 0{\cdot}5$, $p_2 = q = \alpha = 0$ (i.e., example 2 of reference [8]), studied by simulation; all of them are non-linear normal phase manifolds obtained by projection of the trajectory for different initial conditions, energy and base points: (e) [base point, initial condition, energy] = [first component point, first NNM, 0·1528]; (f) [base point, initial condition, energy] = [first component point, second NNM, 3·393]; (g) [base point, initial condition, energy] = [abstract point, first-NNM, 0·1528]; (h) [base point, initial condition, energy] = [abstract point, second NNM, 3·393].

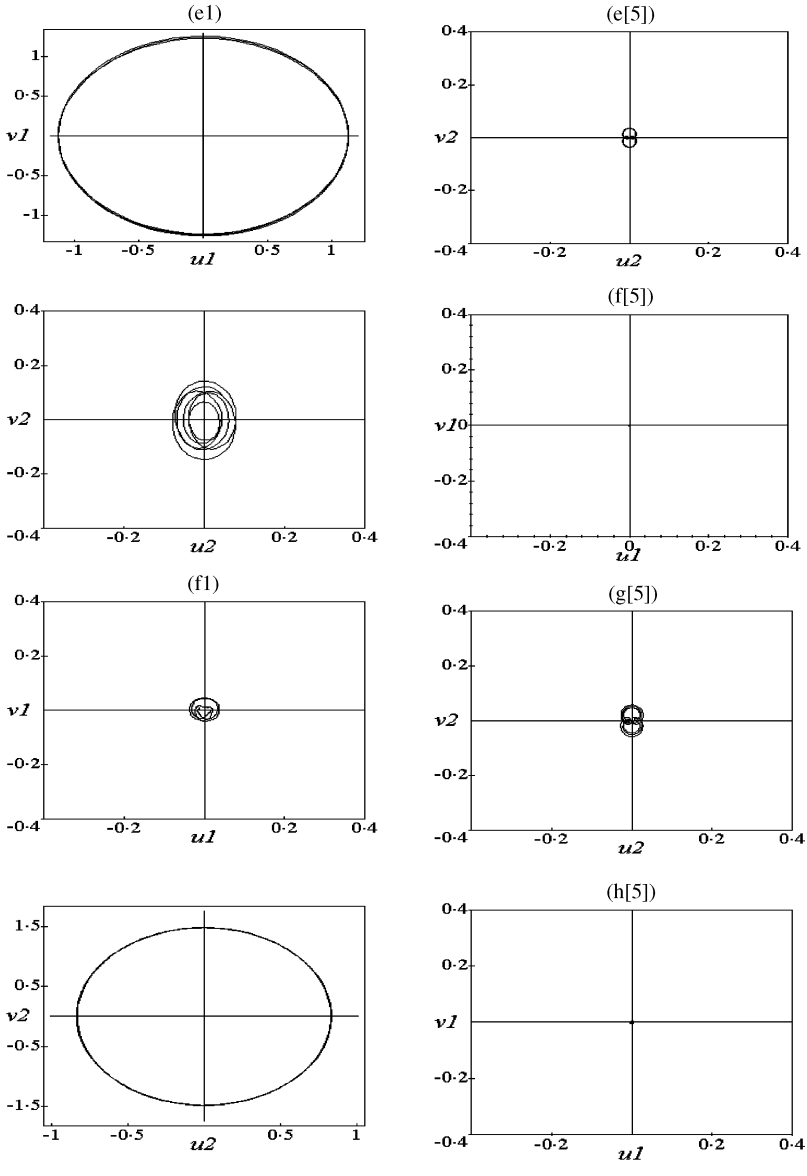

Figure 3. Same as Figure 2 above except that (e1) [base point, initial condition, energy] = [second-component point, first-NNM, 0·1528]; (f1) [base point, initial condition, energy] = [second-component point, second NNM, 3·393]; for (e[5]), (f[5], (g[5]) and (h[5]), the NNMs are approximated to fifth order; (e[5]) the second non-linear normal phase manifold, [base point, initial condition, energy] = [first-component point, first-NNM, 0·1581]; (f[5]) the first non-linear normal phase manifold, [base point, initial condition, energy] = [first-component point, second NNM, 3·464]; (g[5]) the second non-linear normal phase manifold, [base point, initial condition, energy] = [abstract point, first NNM, 0·1581]; (h[5]) the first non-linear normal phase manifold, [base point, initial condition, energy] = [abstract point, second NNM, 3·464].

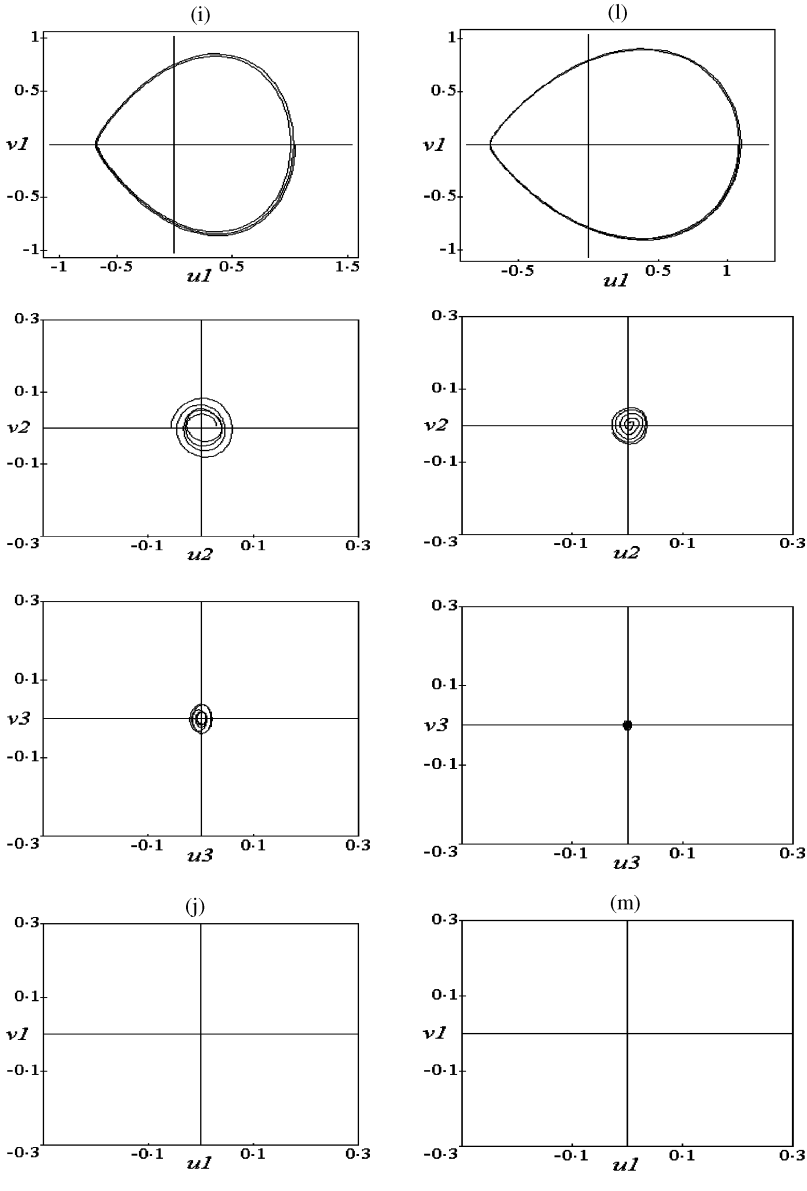

Figure 4. Effects of the base points on the invariance of the non-linear normal modal subspaces of the FPU-2 model (7) with the parameters $\alpha = 0{\cdot}5$, studied by simulation; all of them are non-linear normal phase manifolds obtained by the projection of the trajectory for different initial conditions, energy and base points: (i) [base point, initial condition, energy] = [first component point, first-NNM, $-1{\cdot}13$]; (j) [base point, initial condition, energy] = [first-component point, second-NNM, $1{\cdot}833$]; (k) [base point, initial condition, energy] = [first-component point, third-NNM, $6{\cdot}186$]; (1) [base point, initial condition, energy] = [abstract point, first NNM, $-1{\cdot}13$]; (m) [base point, initial condition, energy] = [abstract point, second-NNM, $1{\cdot}833$]; (n) [base point, initial condition, energy] = [abstract point, third NNM, $6{\cdot}186$].

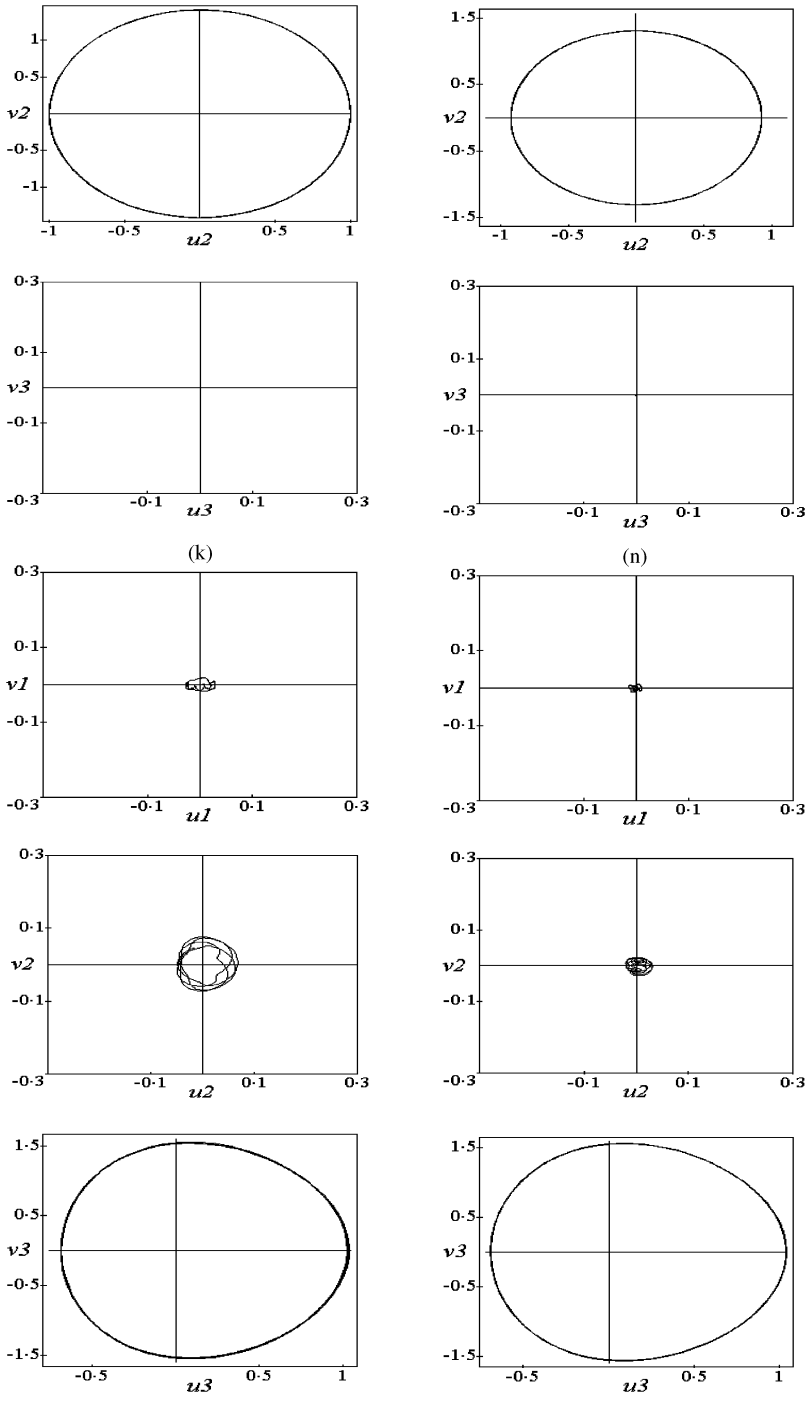

Figure 4. Continued.

REFERENCES

1. R. H. ROSENBERG 1966 *Advances in Applied Mechanics* **9**, 155–242. On non-linear vibrations of systems with many degrees of freedom.
2. T. K. CAUGHEY, A. VAKAKIS and V. SIVO 1990 *International Journal of Non-linear Mechanics* **25**, 521–533. Analytical study of similar normal modes and their bifurcations in a class of strongly non-linear systems.
3. A. VAKAKIS and R. H. RAND 1992 *International Journal of Non-linear Mechanics* **27**, 861–874. Normal modes and global dynamics of a two-degree-of-freedom non-linear system—I. Low energies.
4. A. VAKAKIS and R. H. RAND 1992 *International Journal of Non-linear Mechanics* **27**, 875–888. Normal modes and global dynamics of a two-degree-of-freedom non-linear system—II. High energies.
5. A. F. VAKAKIS and T. K. CAUGHEY 1992 *American Society of Mechanical Engineers Journal of Applied Mechanics* **59**, 418–424. A theorem on the exact nonsimilar steady-state motions of a non-linear oscillator.
6. A. F. VAKAKIS 1992 *Journal of Sound and Vibration* **158**, 341–361. Non-similar normal oscillations in a strongly non-linear discrete system.
7. T. A. NAYFEY, E. EMACI and A. F. VAKAKIS 1997 *American Institute of Aeronautics and Astronautics Journal* **35**, 1378–1386. Application of non-linear localization to the optimization of a vibration isolation system.
8. S. SHAW and C. PIERRE 1993 *Journal of Sound and Vibration* **164**, 85–124. Normal modes for non-linear vibratory systems.
9. J. XU, Q. S. LU and K. L. HUANG 1998 *Applied Mathematics and Mechanics* (*English Ed.*) **19**, 1167–1177. Non-linear normal modes and their superposition in a two degrees of freedom asymmetric system with cubic non-linearities.
10. X. H. ZHANG 2000 *Proceedings of International Conference on Advanced Problems in Vibration Theory and Applications* (*ICAPV* 2000) 518–522. Beijing: Science Press. Applications of the NNM method to FPU model.
11. T. D. BUTON and W. RHEE 2000 *Journal of Vibration and Control* **6**, 531–556. On the reduction of non-linear structural dynamics models.
12. G. CHECHIN, V. P. SAKHNENKO, H. T. STOKES, A. D. SMITH and D. M. HATCH 2000 *International Journal of Non-linear Mechanics* **35**, 497–513. Non-linear normal modes for systems with discrete symmetry.
13. S. SHAW and C. PIERRE 1994 *Journal of Sound and Vibration* **169**, 319–347. Normal modes of vibration for non-linear continuous systems.
14. A. H. NAYFEY and S. A. NAYFEY 1994 *Journal of Vibration and Acoustics* **116**, 129–136. On non-linear modes of continuous systems.
15. M. E. KING and A. F. VAKAKIS 1994 *American Society of Mechanical Engineers Journal of Vibration and Acoustics* **116**, 332–340. An energy-based formulation for computing non-linear normal modes in undamped continuous systems.
16. E. FERMI, J. PASTA and S. ULAM 1965 E. Fermi Collected Papers, Vol. II, 978–988. Chicago: University of Chicago Press. Studies of non-linear problems.

APPENDIX A: THE COEFFICIENTS OF THE NON-LINEAR NORMAL MODES

When the parameters take the values $\alpha = 0$, $p_1 = g$, $p_2 = 0$, $q = 0$, the coefficients of the NNMs are as follows (with abstract base points)

Mode I

$$a_{2i} = a_{3i} = a_{4i} = a_{5i} = a_{7i} = a_{9i} = b_{1i} = b_{3i} = b_{4i} = b_{5i} = b_{6i} = b_{8i} = 0, \quad i = 1,2,$$

$$a_{61} = -a_{62} = -\frac{\sqrt{2}g(k-3)}{16k(k-4)}, \quad a_{81} = -a_{82} = b_{91} = -b_{92} = \frac{3\sqrt{2}g}{16k(k-4)},$$

$$b_{71} = -b_{72} = -\frac{3\sqrt{2}g(k-1)}{16k(k-4)}.$$

Mode II

$$a_{2i} = a_{3i} = a_{4i} = a_{5i} = a_{7i} = a_{9i} = b_{1i} = b_{3i} = b_{4i} = b_{5i} = b_{6i} = b_{8i} = 0, \quad i = 1, 2,$$

$$a_{61} = a_{62} = \frac{\sqrt{2}g(7k+3)}{16k(9k+4)}, \quad a_{81} = a_{82} = b_{91} = b_{92} = \frac{3\sqrt{2}g}{16k(9k+4)},$$

$$b_{71} = b_{72} = \frac{3\sqrt{2}g(3k+1)}{16k(9k+4)}.$$

APPENDIX B: THE COEFFICIENTS C_{3i} OF THE NON-LINEAR NORMAL MODAL EQUATIONS

$$\begin{aligned}
c_{3i} = (&-44\alpha^2 k p_1 \delta + 8\alpha^2 k q \delta^2 - 78\alpha^2 k p_2 \delta^2 + 42\alpha^3 k \delta p_2 + 26\alpha^2 k p_1 \delta^2 \\
&- 208\alpha^2 k q \delta + 148\alpha^2 k \delta p_2 + 416\alpha k \delta^3 q + 96\alpha k \delta p_2 - 10\alpha k \delta^3 p_1 \\
&- 96\alpha k p_1 \delta - 800\alpha k q \delta^2 - 152\alpha k p_2 \delta^2 + 62\alpha k \delta^3 p_2 + 384\alpha k \delta q + 56\alpha k p_1 \delta^2 \\
&- 168\alpha^3 k \delta q + 10\alpha^3 k p_1 \delta - 416 q \delta^4 - 128 q \delta^2 - 17\alpha^2 \delta^3 p_1 - 32\alpha^4 \delta q - 8\alpha^4 \delta p_2 \\
&+ 8\alpha^4 p_1 \delta - 48\alpha^3 k p_2 + 192\alpha^3 k q - 48\alpha^3 k p_1 - 208\alpha q \delta^4 - 26\alpha \delta^3 p_1 + 28\alpha p_1 \delta^2 \\
&- 192\alpha \mathrm{q} \delta^2 + 78\alpha \delta^3 p_2 - 76\alpha \delta^2 p_2 - 31\alpha \delta^4 p_2 + 32\alpha \delta p_2 - 32\alpha \delta p_1 + 400\alpha \delta^3 q \\
&+ 5\alpha \delta^4 p_1 - 8\alpha^4 k p_2 + 32\alpha^4 k q - 8\alpha^4 k p_1 - 48\alpha^2 k p_1 + 192\alpha^2 k q - 48\alpha^2 k p_2 \\
&+ 832\delta^3 k q - 48\delta^2 k p_2 + 52\delta^3 k p_1 + 52\delta^3 k p_2 - 18\delta^4 k p_2 + 256\delta k q - 18\delta^4 k p_1 \\
&- 48\delta^2 k p_1 - 800\delta^2 k q - 288\delta^4 k q + 188\alpha^3 \delta^2 q + 26\alpha^3 \delta p_2 - 9\alpha^3 \delta^2 p_1 \\
&- 296\alpha^3 \delta q + 26\alpha^3 \delta p_1 - 9\alpha^3 \delta^2 p_2 - 76\alpha^2 \delta^3 q + 26\alpha^2 \delta^2 p_1 - 22\alpha^2 \delta p_1 \\
&- 78\alpha^2 \delta^2 p_2 + 74\alpha^2 \delta p_2 - 296\alpha^2 \delta q + 312\alpha^2 \delta^2 q + 35\alpha^2 \delta^3 p_2 - 16\delta^2 p_1 \\
&- 26\delta^4 p_1 - 26\delta^4 p_2 + 26\delta^3 p_1 + 9\delta^5 p_1 + 144\delta^5 q + 9\delta^5 p_2 - 16\delta^2 p_2 - 24\alpha^3 p_2 \\
&- 24\alpha^3 p_1 + 96\alpha^3 q + 400\delta^3 q + 64\alpha^2 q + 26\delta^3 p_2 - 16\alpha^2 p_2 - 16\alpha^2 p_1 + 4\alpha^5 p_2 \\
&+ 4\alpha^5 p_1 - 16\alpha^5 q)/(4\delta^2(-26\alpha\delta k + 48\alpha k - 4\alpha^3 + 18\delta^2 k - 52\delta k + 48k + 8\alpha^2 k \\
&- 9\delta^3 + 26\delta^2 - 26\delta + 16 + 13\alpha\delta^2 - 26\alpha\delta + 24\alpha)).
\end{aligned}$$

The above expression is used for the computation of the coefficient c_{31} of the first-NNM equation in equation (10). The expression for the coefficient c_{32} of the second-NNM equation is the same as the one above except that the parameter δ should be replaced by "$-\delta$".